CONVEX-TRANSITIVE BANACH SPACES AND THEIR HYPERPLANES

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Academic dissertation

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This humble work is dedicated to the fond memory of my grandmother Aili Haverinen and of my friend, Tuomo Malinen.

Cádiz, May 2008
J.T.

List of included articles

This dissertation consists of an introductory part and the following publications:

[A] J. Talponen, *Asymptotically transitive Banach spaces*, in Banach Spaces and their Applications in Analysis, de Gruyter (2007), 423-438.

[B] J. Talponen, *Convex-transitive characterizations of Hilbert spaces*, Proceedings of the Royal Society of Edinburgh: Section A (to appear).

[C] J. Talponen, *Convex-transitivity and function spaces*, Journal of Mathematical Analysis and Applications (to appear).

In the introductory part these articles will be referred to as [A], [B] and [C], whereas other references will be numbered as [1], [2], [3], etc.

Introduction

In this Ph.D. thesis we study symmetries of Banach spaces. The concept of symmetry can of course be understood in several ways. Here it is related to *rotations*, that is, linear onto isometries $T\colon \mathrm{X} \to \mathrm{X}$, where X is a Banach space. Recall that a map $T\colon \mathrm{X} \to \mathrm{X}$ is an isometry if $||Tx - Ty|| = ||x - y||$ for all $x, y \in \mathrm{X}$. Observe that an isometry is necessarily injective and hence the rotations are bijective. Recall also that the concept of a rotation admits a formally weaker definition, since the Mazur-Ulam theorem states that each onto isometry $T\colon \mathrm{X} \to \mathrm{X}$ such that $T(0) = 0$ is already linear.

The study of rotations is a classical topic in the theory of Banach spaces. For example, a complete classification of the rotations of ℓ^p and L^p, for $1 \leq p \leq \infty$, was indicated already in Banach's monograph [4].

It is a natural question in Banach space theory, given a pair of Banach spaces X and Y, to decide whether there is a bijective linear isometry $\mathrm{X} \to \mathrm{Y}$. Likewise, given a Banach space X, it is a classical and equally important problem to find all the pairs of points $x, y \in \mathrm{X}$ for which there exists a rotation T of X such that $T(x) = y$. This work is mainly motivated by the following Banach-Mazur rotation problem, which appeared in Banach's monograph: *Suppose that* X *is a separable Banach space and for each* $x, y \in \mathrm{X}$ *with* $||x|| = ||y|| = 1$ *there is a rotation* $T\colon \mathrm{X} \to \mathrm{X}$ *such that* $T(x) = y$. *Does it follow that there exists a linear isometry from* X *onto* ℓ^2*?*

In this thesis we will concentrate on some contemporary variations of this problem. These variations also include almost isometric versions, where isometric conditions are replaced by conditions, which allow small distortions. We will also investigate rotations of Banach lattices and vector-valued function spaces.

Intuitively, a rotation is a transformation that faithfully respects the metric linear structure of the space. To elaborate on the role of symmetries in this thesis, we are interested in pairs (x, y) of points in a Banach space X such that x and y are in a symmetric position or *indistinguishable* from each other by using the linear isometric information available for the space. More precisely, we consider x and y indistinguishable if there is a rotation T of X such that $T(x) = y$, and otherwise we call x and y *distinguishable*. For example, the origin $0 \in \mathrm{X}$ is distinguishable from any point $x \in \mathrm{X}$ with $x \neq 0$, and more generally, if $x, y \in \mathrm{X}$ are such that $||x|| \neq ||y||$, then x and y are distinguishable. Thus, when studying such symmetries of a Banach space it is only relevant to look at the points having a given fixed norm. Hence we will often restrict ourselves to the *unit sphere*

$$\mathbf{S}_{\mathrm{X}} = \{x \in \mathrm{X} : \ ||x|| = 1\}$$

of X. Another important object for us will be the *closed unit ball* of X denoted by

$$\mathbf{B}_{\mathrm{X}} = \{x \in \mathrm{X} : \ ||x|| \leq 1\},$$

which, on the other hand, is the intersection of all the convex subsets of X containing $\mathbf{S}_{\mathrm{X}}$.

The above naive examples about distinguishable pairs illustrate the following tautological but important principle. If $x, y \in \mathbf{S}_{\mathrm{X}}$ are indistinguishable, then x and y satisfy the same isometrically invariant properties. For example, if $T\colon \mathrm{X} \to \mathrm{X}$ is a rotation such that $T(x) = y$, then x is an extreme point of $\mathbf{B}_{\mathrm{X}}$ if and only if y is such. In this way the geometric properties of Banach spaces are often relevant in analyzing the rotations, and more generally, linear isometries between Banach spaces.

The simple analysis we conducted on the equivalence classes of mutually indistinguishable points cannot be taken further as easily as above. One reason is that there exist Banach spaces X such that each pair of points $x, y \in \mathbf{S}_{\mathrm{X}}$ is mutually indistinguishable. In fact the Euclidean space $\mathbb{R}^n$, for $n \in \mathbb{N}$, in its standard norm satisfies this; intuitively speaking, the Euclidean unit ball is perfectly round.

Let us recall why all pairs of points on the unit sphere $\mathbf{S}_{\mathrm{H}}$ of a real Hilbert space H are indistinguishable. The claim is easy if $\dim(\mathrm{H}) \leq 2$, because then H can be identified isometrically and $\mathbb{R}$-linearly either with $\{0\}, \mathbb{R}$ or $\mathbb{C}$. In the last two cases the desired rotation has the form $x \mapsto \lambda x$ for a suitable scalar λ of modulus 1. Let us consider the case where $\dim(\mathrm{H}) \geq 3$. Take $x, y \in \mathbf{S}_{\mathrm{H}},\ x \neq \pm y$, and let $A \subset H$ be the 2-dimensional subspace spanned by x and y. Let $P\colon \mathrm{H} \to A$ be the orthogonal projection onto A. By using the 2-dimensional case, let $T\colon A \to A$ be a rotation which maps $x \mapsto y$. Then the mapping $S = (\mathrm{I} - P) + T \circ P\colon \mathrm{H} \to \mathrm{H}$ is a rotation which maps $x \mapsto y$. Indeed, for any $z \in \mathrm{H}$ it holds that

$$||z||^2 = ||(\mathrm{I} - P)z||^2 + ||Pz||^2 = ||(\mathrm{I} - P)z||^2 + ||(T \circ P)z||^2,$$

and this completes the argument.

We will mainly concentrate on Banach spaces over the real field in this thesis, so in what follows, each Banach space is assumed to be real, unless otherwise stated. Next we will introduce some further basic concepts. We denote by

$$\mathcal{G}_{\mathrm{X}} = \{T\colon \mathrm{X} \to \mathrm{X} \mid T \text{ is a rotation}\}$$

the *rotation group* of X. Here the composition of the maps is the group operation. For $x \in \mathbf{S}_{\mathrm{X}}$ the *orbit of* x corresponding to $\mathcal{G}_{\mathrm{X}}$ is defined as

$$\mathcal{G}(x) = \mathcal{G}_{\mathrm{X}}(x) = \{T(x) |\ T \in \mathcal{G}_{\mathrm{X}}\}.$$

For example, if $1 \leq p \leq \infty,\ p \neq 2$, then each rotation $T \in \mathcal{G}_{\ell^p}$ has the form

$$T((x_n)_{n\in\mathbb{N}}) = (\theta_n x_{\pi(n)})_{n\in\mathbb{N}}, \quad (x_n)_{n\in\mathbb{N}} \in \ell^p, \tag{1}$$

where $\theta_n \in \{-1, 1\}$ for $n \in \mathbb{N}$ and $\pi\colon \mathbb{N} \to \mathbb{N}$ is a permutation (see e.g. [4, p.175-179] or [24, 2.f.14]). By letting $(e_n)_{n\in\mathbb{N}}$ be the standard unit vectors of ℓ^p, this yields that

$$\mathcal{G}_{\ell^p}(e_n) = \{\pm e_m |\ m \in \mathbb{N}\} \quad \text{for } n \in \mathbb{N}. \tag{2}$$

Next, we collect the most important transitivity concepts appearing in this thesis. Let $(\mathrm{X}, ||\cdot||)$ be a Banach space.

(i) $(\mathrm{X}, ||\cdot||)$ is *transitive* if $\mathcal{G}_{\mathrm{X}}(x) = \mathbf{S}_{\mathrm{X}}$ for each $x \in \mathbf{S}_{\mathrm{X}}$.
(ii) $(\mathrm{X}, ||\cdot||)$ is *almost transitive* if $\overline{\mathcal{G}_{\mathrm{X}}(x)} = \mathbf{S}_{\mathrm{X}}$ for each $x \in \mathbf{S}_{\mathrm{X}}$.
(iii) $x \in \mathbf{S}_{\mathrm{X}}$ is a *big point* if $\overline{\mathrm{conv}}(\mathcal{G}_{\mathrm{X}}(x)) = \mathbf{B}_{\mathrm{X}}$.
(iv) $(\mathrm{X}, ||\cdot||)$ is *convex-transitive* if each $x \in \mathbf{S}_{\mathrm{X}}$ is a big point.

Recall also that

(v) $(\mathrm{X}, ||\cdot||)$ is *maximally normed* if for each equivalent norm $|||\cdot||| \sim ||\cdot||$ such that $\mathcal{G}_{(\mathrm{X},||\cdot||)} \subset \mathcal{G}_{(\mathrm{X},|||\cdot|||)}$ it holds that $\mathcal{G}_{(\mathrm{X},||\cdot||)} = \mathcal{G}_{(\mathrm{X},|||\cdot|||)}$.

It should be noted that in the literature these properties of the spaces are sometimes referred to as being properties of the corresponding norms.

Observe that transitivity implies almost transitivity, which in turn implies convex-transitivity. To complete the chain of implications, let us mention that convex-transitivity implies the maximality of the norm, see e.g. [15, Thm. 12.4.1.].

It follows immediately from (2) that ℓ^p is not transitive for $p \neq 2$, since $2^{-\frac{1}{p}}(e_1 + e_2) \notin \mathcal{G}_{\ell^p}(e_1)$. As the condition of transitivity appears to be quite restrictive, one might ask if Hilbert spaces are in some sense typical in the class of transitive Banach spaces. This type of question may be considered as old as Banach space theory itself. The above-mentioned *rotation problem* is the following question appearing on page 242 of Banach's classical monograph [4] from 1932:
...La forme générale (15) *des rotations dans* (L^2)*, établie p.174, (d'ailleurs connue dépuis longtemps), montre que pour tout couple d'éléments* x *et* y *à la norme* 1 *il y existe une rotation autour de* θ *qui transforme* x *en* y*. M. S. Mazur a posé la question si tout espace du type* (B) *séparable, ayant une infinité de dimensions et jouissant de cette propriété est isométrique avec* (L^2)*.*
This problem can be reformulated succinctly as follows:
Is every transitive separable Banach space isometric to $L^2(0, 1)$*?*
This problem is still unsolved and a substantial branch of Banach space theory has evolved around it. We refer to [7] for a contemporary survey of the field (see also [28] for classical results). The monographs [14] and [15] include also very recent advances.

Separability and transitivity are essential in the rotation problem. In fact, Pelczynski and Rolewicz [26] proved in 1962 that if Γ is an uncountable set, then the space $\sum_\Gamma L^p(0, 1)$, where the summation is understood in the ℓ^p-sense, is transitive (see [28]). They also proved that $L^p(0, 1)$ is almost transitive for $1 \leq p < \infty$ and that $L^\infty(0,1)$ is convex-transitive. On the other hand, $L^p(0,1)$ is transitive only if $p = 2$, see e.g. [28].

In the finite-dimensional case there are stronger results available due to Auerbach [3] (1935): If $(\mathrm{X}, ||\cdot||)$ is a finite-dimensional normed space, then there is an inner product $(\cdot,\cdot)$ on X, which induces an equivalent norm on X, such that $(Tx, Ty) = (x, y)$ for $x, y \in \mathrm{X}$ and $T \in \mathcal{G}_{(\mathrm{X},||\cdot||)}$, see [3]. Consequently, if $||\cdot||$ above is a maximal norm, then X is in fact isometric to an inner product space.

Convex-transitivity

Since convex-transitivity is the most important theme in this dissertation, we will next discuss this property in more detail to provide additional background information.

We first recall that Pelczynski and Rolewicz [26] (see also [28]) discovered that the space $C(K)$ of real-valued continuous functions on K, where K is the Cantor set, is convex-transitive but not almost transitive with respect to the standard $||\cdot||_\infty$ norm. Recall that $C_0(L)$ is the $||\cdot||_\infty$-normed space of real-valued continuous functions f on L such that for each $\epsilon > 0$ there is a compact subset $K \subset L$ with $|f(t)| < \epsilon$ for $t \in L \setminus K$. The space L is totally disconnected if each connected component of L is a singleton. Wood [30] observed that for a locally compact space L the space $C_0(L)$ is convex-transitive if and only if L is totally disconnected, and for every probability measure μ on L and $t \in L$ there exists a net $\{\gamma_\alpha\}_\alpha$ of homeomorphisms on L such that the net $\{\mu \circ \gamma_\alpha\}_\alpha$ is ω^*-convergent to the Dirac measure δ_t. The above measure $\mu \circ \gamma_\alpha$ is given by $\mu \circ \gamma_\alpha(A) = \mu(\gamma_\alpha(A))$ for $A \subset L$ such that $\gamma_\alpha(A)$ is μ-measurable.

Cowie [11] showed that convex-transitivity of $(\mathrm{X}, ||\cdot||)$ is characterized by the condition that for every equivalent norm $|||\cdot|||$ on X such that $\mathcal{G}_{(\mathrm{X},||\cdot||)} \subset \mathcal{G}_{(\mathrm{X},|||\cdot|||)}$ there is $c > 0$ such that $|||\cdot||| = c||\cdot||$. In [7] the following equivalence is mentioned as a folklore result: X is convex-transitive if and only if $\overline{\mathrm{conv}}^{\omega^*}(\{T^* f \mid T \in \mathcal{G}_\mathrm{X}\}) = \mathbf{B}_{\mathrm{X}^*}$ for every $f \in \mathbf{S}_{\mathrm{X}^*}$.

We will mainly be working with convex-transitive spaces in papers [B] and [C].

Decompositions of orthogonal type

Next we will recall some relevant concepts, which resemble orthogonal decompositions of Hilbert spaces. Suppose that X is a Banach space, $M \subset \mathrm{X}$ a closed subspace and $1 \leq p \leq \infty$. The subspace M is said to be an ℓ^p-*summand* of X if there is a (surjective) linear projection $P\colon \mathrm{X} \to M$ such that, for every $x \in \mathrm{X}$ we have

$$\begin{aligned} ||x||^p &= ||P(x)||^p + ||x - P(x)||^p \quad (1 \leq p < \infty) \\ ||x|| &= \max(||P(x)||, ||x - P(x)||) \quad (p = \infty). \end{aligned} \tag{3}$$

A weaker condition than M being a ℓ^p-summand is that P above satisfies $\mathrm{I} - 2P \in \mathcal{G}_{\mathrm{X}}$. In such a case M is said to be an *isometric reflection subspace*. It follows from the triangle inequality that if the range of P is non-trivial, then P is a *bicontractive* projection, i.e. $||P|| = ||\mathrm{I} - P|| = 1$, since $-P = \frac{\mathrm{I}-2P-\mathrm{I}}{2}$ and $\mathrm{I} - P = \frac{\mathrm{I}-2P+\mathrm{I}}{2}$.

If $f \in \mathbf{S}_{\mathrm{X}^*}$ and $x \in \mathbf{S}_{\mathrm{X}}$ are such that $f(x) = 1$, then the operator $\mathrm{X} \to \mathrm{X}$ given by $y \mapsto f(y)x$ is a norm-one projection, which is denoted by $f \otimes x$. Recall that for a non-empty set $A \subset \mathrm{X}$ a map $P_A \colon \mathrm{X} \to A$ is said to be a *closest point selection*, if $||x - P_A(x)|| = \mathrm{dist}(x, A)$ for all $x \in \mathrm{X}$. In fact, a linear projection $P \colon \mathrm{X} \to M$ is a closest point selection if and only if $||\mathrm{I} - P|| = 1$, see [2, Lm.13.1]. Hence each bicontractive projection is also a closest point selection. In papers [A] and [B] we will frequently apply the following useful characterization of Hilbert spaces [2, 13.4']:
Let X *be a Banach space with* $\dim(\mathrm{X}) \geq 3$*. Then each* 1*-dimensional subspace* $L \subset \mathrm{X}$ *admits a linear closest point selection* P_L *if and only if* X *is isometrically a Hilbert space.*

Becerra and Rodriguez [5] observed that if a convex-transitive space X contains a 1-dimensional ℓ^p-summand for some $1 \leq p \leq \infty$, then X is isometrically a Hilbert space. Randrianantoanina proved in [27] that an almost transitive Banach space, which contains a 1-complemented 1-codimensional subspace, is a Hilbert space. This result was also observed independently by Cabello in his unpublished Ph.D. thesis [8].

Here we will generalize this result in various directions. A considerable part of this thesis, namely papers [A] and [B], is devoted to the almost isometric and convex-transitive versions of Randrianantoanina's above-mentioned result. In [27] she posed the following question:
Is each convex-transitive Banach space, which admits a 1*-complemented* 1*-codimensional subspace in fact a Hilbert space?*
This is one of the main problems motivating this thesis and it is treated in paper [B]. The above problem has not been solved even for real convex-transitive spaces, which admit a 1-dimensional isometric reflection subspace. However, in [B] it turns out that under very weak geometric assumptions about the norm the answer to Randrianantoanina's question is affirmative.

Observe that any non-trivial Banach space over the complex field admits plenty of rotations obtained by the multiplication with scalars of modulus 1. Inspired by the results of Kalton and Wood [21], Becerra and Rodriguez [6] proved that if X is a complex Banach space such that $\mathrm{X}_{\mathbb{R}}$ is convex-transitive and admits a 1-dimensional isometric reflection subspace, then X is a (complex) Hilbert space. Above $\mathrm{X}_{\mathbb{R}}$ is X restricted to multiplication by only real scalars.

Rotations of function spaces

It is a natural idea to use the special structure of function spaces to analyze their rotations. In papers [B] and [C] we will study rotations of some Banach lattices and function spaces, respectively. In the latter paper we work in both real-valued and vector-valued settings.

In what follows (Ω, Σ, μ) is a positive measure space. Given $p \in [1, \infty)$ satisfying $p \neq 2$, suppose that X is a separable Banach space such that X is not isometrically isomorphic to $\mathrm{Y} \oplus_p \mathrm{Z}$ for any non-trivial Y and Z. Then the rotations of the Bochner space $L^p(\mu, \mathrm{X})$ have an expected form. In fact, the following formula defines, in a sense to be made precise below, a rotation $T \in \mathcal{G}_{L^p(\mu,\mathrm{X})}$:

$$(4) \quad T(f)(s) = \sigma(s) \left(\frac{\mathrm{d}(\mu \circ \phi^{-1})}{\mathrm{d}\mu}(s) \right)^{\frac{1}{p}} (f \circ \phi^{-1})(s) \quad \text{for } \mu\text{−a.e. } s \in \Omega.$$

Above, the map $\phi \colon \Sigma\backslash_\mu \to \Sigma\backslash_\mu$ is a Boolean isomorphism, $\frac{\mathrm{d}(\mu \circ \phi^{-1})}{\mathrm{d}\mu}$ is the Radon-Nikodym derivative, and $\sigma \colon \Omega \to \mathcal{G}_{\mathrm{X}}$ is a strongly measurable map. Let us briefly recall these notions. Here we are following the terminology and notations used in [15], which we also refer to for further details.

The Boolean algebra $\Sigma\backslash_\mu$ is obtained from Σ as a quotient by identifying all μ-null sets with $\emptyset$. The requirement that ϕ is a Boolean isomorphism means that ϕ is a bijection such that $\phi(A \vee B) = \phi(A) \vee \phi(B)$ and $\phi(A^c) = \phi(A)^c$ for all $A, B \in \Sigma\backslash_\mu$. Recall that $h \colon \Omega \to \mathrm{X}$ is said to be Bochner measurable if there exists a sequence $(h_n)_{n \in \mathbb{N}}$ of measurable simple functions $\Omega \to \mathrm{X}$ such that $h_n \to h$ μ-a.e. as $n \to \infty$. Measurable simple functions $\Omega \to \mathrm{X}$ are of the form $f = \sum_{k=1}^{n} \chi_{A_k} x_k$, where $x_1, \ldots, x_n \in \mathrm{X}$ and $A_1, \ldots, A_n \in \Sigma$. The strong measurability of σ means here that the mapping $\sigma(\cdot)(x)$ is Bochner measurable for each $x \in \mathrm{X}$.

The role of $\frac{\mathrm{d}(\mu \circ \phi^{-1})}{\mathrm{d}\mu}$ in (4) is to compensate for the distortion of the measure under the transformation ϕ. This matter is best explained by the following change of variable identity:

$$\int_{\phi(A)} |f \circ \phi^{-1}(t)|^p \, \frac{\mathrm{d}(\mu \circ \phi^{-1})}{\mathrm{d}\mu}(t) \ \ \mathrm{d}\mu(t) = \int_A |f(t)|^p \, \mathrm{d}\mu(t) \quad \text{for } A \in \Sigma.$$

Next, we will describe in which sense (4) defines a rotation. Given a simple measurable function f, the function $T(f)$ is properly defined by (4), since $f \circ \phi^{-1}$ is a Bochner measurable simple function. Similarly $T^{-1}(f)$ is well-defined and Bochner measurable. By using the fact that the measurable simple functions are *norm-dense* in $L^p(\mu, \mathrm{X})$, one obtains that (4) extends to a rotation of $L^p(\mu, \mathrm{X})$. If X is separable and $\mathrm{X} \neq \mathrm{Y} \oplus_p \mathrm{Z}$ for any non-trivial subspaces $\mathrm{Y}, \mathrm{Z} \subset \mathrm{X}$, then the converse holds also: Each $T \in \mathcal{G}_{L^p(\mu,\mathrm{X})}$ can be represented by (4) in the above sense, see [18], [17].

We also note that if $\mathrm{X} = \mathrm{Y} \oplus_p \mathrm{Z}$ for some non-trivial subspaces Y and Z, then the representation (4) fails. For example, let μ be the probability measure on $\{0,1\}$ given by $\mu(0) = \mu(1) = \frac{1}{2}$. Write each $f \in L^p(\mu, \mathrm{X})$ as $f(0) = y_0 + z_0,\ f(1) = y_1 + z_1 \in \mathrm{Y} \oplus_p \mathrm{Z}$. Then $(y_0 + z_0, y_1 + z_1) \mapsto (y_0 + z_1, y_1 + z_0)$ defines a rotation on $L^p(\mu, \mathrm{X})$, which is incompatible with formula (4).

Recall finally that it was shown by Greim, Jamison and Kaminska [19] that $L^p(\mu, \mathrm{X})$ is almost transitive whenever μ is a non-atomic positive measure, $1 \leq p < \infty$, and X is almost transitive.

Geometry of the norm

In [B] the geometry of the norm plays a central part. We denote the weak topology by ω. Let $C \subset \mathrm{X}$ be a closed convex subset. A point $x \in C$ is called *ω-exposed* (resp. *strongly exposed*) if there is $f \in \mathbf{S}_{\mathrm{X}^*},\ f(x) = ||x||$, which satisfies the following: Whenever $(x_n) \subset C$ is a sequence such that $f(x_n) \to ||x||$ as $n \to \infty$, then $x_n \xrightarrow{\omega} x$ (resp. $x_n \xrightarrow{||\cdot||} x$) as $n \to \infty$. These exposedness concepts can be regarded as forms of strong convexity of convex bodies.

A point $x \in \mathbf{S}_{\mathrm{X}}$ is called Gateaux (resp. Fréchet) smooth if there is a unique functional $f \in \mathbf{S}_{\mathrm{X}^*},\ f(x) = 1$, such that the following holds: For any sequence $(f_n) \subset \mathbf{B}_{\mathrm{X}^*}$ such that $f_n(x) \to 1$ as $n \to \infty$ it follows that $f_n \xrightarrow{\omega^*} f$ (resp. $f_n \xrightarrow{||\cdot||} f$) as $n \to \infty$. The previous definitions suggest that there exists a duality between many concepts of convexity and smoothness (for more information about this, see e.g. [20], [29]). If τ is a locally convex topology on X, then a point $x \in \mathbf{S}_{\mathrm{X}}$ is τ *locally uniformly rotund*, τ-LUR for short, provided the following condition holds: Whenever $(x_n) \subset \mathbf{B}_{\mathrm{X}}$ is a sequence such that $||x + x_n|| \to 2$ as $n \to \infty$, then $x_n \xrightarrow{\tau} x$ as $n \to \infty$.

Next, we recall two properties of Banach spaces that are weaker than reflexivity. A Banach space X is called *Asplund* if each separable subspace $\mathrm{Y} \subset \mathrm{X}$ has a separable dual Y^*. It is said that X has the *Radon-Nikodym Property* (RNP for short) if each closed convex bounded subset $C \subset \mathrm{X}$ is the closed convex hull of its strongly exposed points. Recall that the Asplund property and the RNP are dual properties in the sense that X is an Asplund space if and only if X^* has the RNP. We refer to the first chapter of [20] for a discussion and an introduction to the geometry of the norm.

Nice geometric properties of a Banach space X tend to accumulate if X has a rich structure of symmetries. By this we mean that if X has some nice geometric property and $\mathcal{G}_{\mathrm{X}}$ is rich (in some sense), then X often has even some stronger geometric properties. For example, a superreflexive almost transitive Banach space is both uniformly convex and uniformly smooth, as was observed by Finet in [13]. More generally, Cabello [10] showed that the Asplund property or the RNP of a convex-transitive Banach space X already implies that X is simultaneously almost transitive, uniformly convex and uniformly smooth. We mention that it is not known whether each almost transitive Banach space, which is isomorphic to a Hilbert space, is in fact *isometric* to a Hilbert space.

Summary of Paper A

In the study of the geometric properties of a Banach space it is often of interest to analyze almost isometric properties instead of isometric ones. Denote $\mathrm{Aut}(\mathrm{X}) = \{T \in L(\mathrm{X}) |\ T \text{ is an isomorphism}\}$. The *Banach-Mazur distance* of mutually isomorphic Banach spaces X and Y is given by

$$d_{BM}(\mathrm{X}, \mathrm{Y}) = \inf\{||T|| \cdot ||T^{-1}|| :\ T \in L(\mathrm{X}, \mathrm{Y}) \text{ isomorphism}\}.$$

The spaces X and Y are said to be *almost isometric* if $d_{BM}(\mathrm{X}, \mathrm{Y}) = 1$. In the same spirit one can formulate properties in an almost isometric sense. For example, in [A] we are particularly interested in the following almost isometric version of almost transitivity: A Banach space X is *asymptotically transitive* if for all $x \in \mathbf{S}_{\mathrm{X}}$ the generalized orbit $\mathcal{O}_{\mathrm{X}}(x)$ of x satisfies $\mathcal{O}_{\mathrm{X}}(x) = \mathbf{S}_{\mathrm{X}}$, where

$$\mathcal{O}_{\mathrm{X}}(x) = \bigcap_{\epsilon>0} \{T(x) |\ T \in \mathrm{Aut}(\mathrm{X}),\ \max(||T||, ||T^{-1}||) \leq 1 + \epsilon\}.$$

The main idea employed in [A] is that certain properties of X that hold almost isometrically can be in fact realized isometrically by using the ω^*-compactness of $\mathbf{B}_{\mathrm{X}^*}$ or the Fréchet-smoothness of X. The next result is an almost isometric generalization of Randrianantoanina's above-mentioned characterization of Hilbert spaces.

Theorem 1 ([A]). *Let* X *be an asymptotically transitive Banach space such that for each* $\epsilon > 0$ *there is* 1*-codimensional* $(1 + \epsilon)$*-complemented subspace* $Z_\epsilon \subset \mathrm{X}$*. Then* X *is in fact isometrically isomorphic to a Hilbert space.*

The following illustrative example suggests an abstract approach to proving the above result. If (X_i) is a sequence of asymptotically transitive Banach spaces and $\mathcal{U}$ is a non-principal ultrafilter on $\mathbb{N}$, then the ultraproduct $\prod_{\mathcal{U}} \mathrm{X}_i$ is transitive. See [A, Remark 2.5] for a sketch of an alternative proof of Theorem 1 by means of ultrapowers.

Denote *closed slices* of the unit ball by $S(\mathbf{B}_{\mathrm{X}}, f, \alpha) = \{x \in \mathbf{B}_{\mathrm{X}} | f(x) \geq \alpha\}$ for $f \in \mathbf{S}_{\mathrm{X}^*},\ 0 \leq \alpha \leq 1$. It is said that B_{X} is *dentable* (resp. B_{X^*} is ω^*-*dentable*) if

$$\inf_{\substack{f \in \mathbf{S}_{\mathrm{X}^*},\\ 0<\alpha<1}} \mathrm{diam}(S(\mathbf{B}_{\mathrm{X}}, f, \alpha)) = 0 \quad (\text{resp.} \inf_{\substack{x \in \mathbf{S}_{\mathrm{X}},\\ 0<\alpha<1}} \mathrm{diam}(S(\mathbf{B}_{\mathrm{X}^*}, x, \alpha)) = 0).$$

It follows that a point $x \in \mathbf{S}_{\mathrm{X}}$ is strongly exposed by $f \in \mathbf{S}_{\mathrm{X}^*}$ if $f(x) = 1$ and $\inf_{0<\alpha<1} \mathrm{diam}(S(\mathbf{B}_{\mathrm{X}}, f, \alpha)) = 0$.

Under weaker transitivity conditions (e.g. in almost transitive and convex-transitive cases) local geometric properties tend to improve to global ones. The following theorem provides an example of this phenomenon in the asymptotically transitive case.

Theorem 2 ([A]). *Let* X *be an asymptotically transitive Banach space. If* $\mathbf{B}_{\mathrm{X}}$ *is dentable or* $\mathbf{B}_{\mathrm{X}^*}$ *is* ω^**-dentable, then* X^* *is also asymptotically transitive, and both* X *and* X^* *are uniformly convex and uniformly smooth.*

In Section 4, the author's favourite part of the paper, information about the classical L^p spaces is deduced by applying the fact that L^p is asymptotically transitive for $p < \infty$. Recall Lamperti's result that in L^p for $1 \leq p < \infty,\ p \neq 2$, it holds that

$$||x+y||_p^p + ||x-y||_p^p = 2(||x||_p^p + ||y||_p^p) \tag{5}$$

if and only if the essential supports $\mathrm{supp}(x)$ and $\mathrm{supp}(y)$ are disjoint (see e.g. [22, p.163]). This means that one can detect in purely isometric terms, whether two given functions of L^p have essentially disjoint supports. As a consequence, rotations preserve bands, pairs of disjointly supported functions, etc.

Let $\mathrm{Y} \subset \mathrm{X}$ be a closed subspace. Recall also that a continuous linear projection $P\colon \mathrm{X} \to \mathrm{Y}$, which satisfies

$$||P|| = \inf\{||Q|| :\ Q\colon \mathrm{X} \to \mathrm{Y} \text{ is a linear projection}\}$$

is called a *minimal* projection (onto Y).

Theorem 3 ([A]). *For* $1 < p < \infty$ *denote*

$$\alpha_p = \max_{t\in[0,1]} (t^{p-1} + (1-t)^{p-1})^{\frac{1}{p}} (t^{p^*-1} + (1-t)^{p^*-1})^{\frac{1}{p^*}} - 1, \quad \text{where } \frac{1}{p} + \frac{1}{p^*} = 1$$

and put $\alpha_1 = \alpha_\infty = 1$. *Let* $1 \leq p \leq \infty$ *and suppose that* $P\colon L^p \to [x]$ *is a linear projection, where* $x \in \mathbf{S}_{L^p}$. *Then the following conditions are equivalent:*

(1) $||P|| = 1$.
(2) $||\mathrm{I} - P|| = 1 + \alpha_p$.
(3) $\mathrm{I} - P\colon L^p \to \mathrm{Ker}(P)$ *is a minimal projection.*

This result is closely related to approximation theory in Banach spaces and the constants α_p above appeared in [16].

Let $\mathbf{1} \in L^p \cap L^{p^*}$ be the unit function for $1 < p < \infty$. Write $M^p = \mathrm{Ker}(\mathbf{1}) \subset L^p$, where we consider $\mathbf{1} \in L^{p^*}$. Similarly we consider $2^{\frac{1}{p^*}} \chi_{[0,2^{-1}]}$ as a functional in L^{p^*}. It is observed in [A] that for any given pair of 1-codimensional subspaces $N, M \subset L^p$ it holds that the spaces $N \oplus_p L^p$ and $M \oplus_p L^p$ are isometric. Moreover, by using the Fréchet smoothness of L^p together with the asymptotical transitivity one obtains that N and M are almost isometric. This fact is the main ingredient in the proof of the following result.

Theorem 4 ([A]). *Let* $1 < p < \infty,\ p \neq 2$, *and suppose that* $Y_1, Y_2 \subset L^p$ *are isometric copies of* ℓ^p. *Then there exist unique linear projections* $P_1\colon L^p \to Y_1,\ P_2\colon L^p \to Y_2$ *such that* $||P_1|| = ||P_2|| = 1$ *and* $\mathrm{Ker}(P_1)$ *and* $\mathrm{Ker}(P_2)$ *are* almost isometric.

Unfortunately, we are unable to provide in [A] an example of an asymptotically transitive space, which is not almost transitive. However, Example 5.3 in [A] provides a space X and a point $x \in \mathbf{S}_{\mathrm{X}}$ such that $\overline{\mathcal{G}_{\mathrm{X}}(x)} \neq \mathcal{O}_{\mathrm{X}}(x)$.

Summary of Paper B

In [B] we study Banach spaces X admitting a 1-dimensional bicontractive projection and a big point. The main question is, which additional geometric conditions related to such a space X guarantee that X is isometric to a Hilbert space. This is related to the above-mentioned question by Randrianantoanina, which asks if each real convex-transitive space with a 1-complemented 1-codimensional subspace is a Hilbert space.

The following two results are typical in paper [B].

Theorem 5 ([B]). *Let* X *be a Banach space such that there exists a bicontractive projection* $P\colon \mathrm{X} \to [u]$*, where* $u \in \mathbf{S}_{\mathrm{X}}$ *is a big point. Assume additionally that one of the following geometric conditions hold:*

(a) *For every sequence* $(x_n) \subset \mathbf{B}_{\mathrm{X}}$ *such that* $||u + x_n|| \to 2$ *as* $n \to \infty$ *it holds that* $P(x_n) \to u$ *as* $n \to \infty$*.*

(b) X^* *is* ω^**-LUR.*

Then X *is in fact a Hilbert space.*

Note that $e_1 \in \ell^1$ is a big point and there exists a bicontractive projection $P\colon \ell^1 \to [e_1]$. Hence ℓ^1 satisfies the assumptions of the above result *except* for the conditions (a) or (b). Since ℓ^1 is not a Hilbert space, it follows that the geometric assumptions cannot be simply removed.

The duality mapping $J\colon \mathbf{S}_{\mathrm{X}} \to \mathcal{P}(\mathbf{S}_{\mathrm{X}^*})$ is the set-valued mapping defined by

$$J(x) = \{f \in \mathbf{S}_{\mathrm{X}^*} | f(x) = 1\}, \quad \text{for } x \in \mathbf{S}_{\mathrm{X}}.$$

Given topological spaces X, Y and a point $x \in X$, a set-valued map $f\colon X \to \mathcal{P}(Y)$ is called *upper semi-continuous* (u.s.c.) at x if for each open set $V \subset Y$, which contains $f(x)$, there exists an open neighbourhood $U \subset X$ of x such that $f(U) \subset V$.

Theorem 6 ([B]). *Let* X *be a convex-transitive Banach space, which admits a bicontractive projection* $P\colon \mathrm{X} \to [u]$*, where* $u \in \mathbf{S}_{\mathrm{X}}$*. Consider* $u \in \mathbf{S}_{\mathrm{X}^{**}}$ *and assume that the following conditions hold:*

(i) *The weakly exposed points of* $\mathbf{B}_{\mathrm{X}}$ *are dense in* $\mathbf{S}_{\mathrm{X}}$*.*

(ii) *The set-valued map* $u \circ J\colon (\mathbf{S}_{\mathrm{X}}, \omega) \to \mathcal{P}([-1,1])$ *is u.s.c. at* u*.*

Then X *is a Hilbert space.*

The geometric conditions (i) and (ii) hold for instance, if X^* is smooth and ω^*-LUR.

The proofs of the above results include a similar trick. Namely, a density reduction, which allows us to pass to a suitable separable subspace, where the argument then proceeds.

Theorem 7 ([B]). *Let* X *be a Banach space which is convex-transitive with respect to a given subgroup of isometries* $\mathcal{G}^0 \subset \mathcal{G}_{\mathrm{X}}$*, and let* $C \subset \mathbf{S}_{\mathrm{X}}$ *be a given norm-dense subset. Assume that* $A \subset \mathrm{X}$ *is a closed subspace with density character* $\mathrm{dens}(A) = \kappa$*. Then there exists a closed subspace* $\mathrm{Y} \subset \mathrm{X}$ *such that*

(1) $A \subset \mathrm{Y}$,
(2) $\mathrm{dens}(\mathrm{Y}) = \kappa$,
(3) Y *is convex-transitive with respect to the subgroup* $\mathcal{G}^0_{\mathrm{Y}} = \{T|_{\mathrm{Y}} |\ T \in \mathcal{G}^0,\ T(\mathrm{Y}) = \mathrm{Y}\} \subset \mathcal{G}_{\mathrm{Y}}$,
(4) $\overline{C \cap \mathbf{S}_{\mathrm{Y}}} = \mathbf{S}_{\mathrm{Y}}$.

The construction of the space Y above applies a back-and-forth type of recursion of countable length. Similar results are also considered at the end of paper [C]. The *density character* of a Banach space X is the least cardinal κ such that there is a dense subsets $A \subset \mathrm{X}$ with $|A| = \kappa$.

Recall Mazur's classical result stating that the smooth points are dense in the unit sphere of a separable Banach space. This fact will be very convenient for us, since the following auxiliary result applies to smooth points and we apply it in combination with Theorem 7.

Lemma 8 ([B]). *Let* X *be a Banach space and let* $(x_n, f_n) \in \mathbf{S}_{\mathrm{X}} \times \mathbf{S}_{\mathrm{X}^*},\ n \in \mathbb{N}$*, be pairs such that* $f_n(x_n) = 1$ *for each* n*. Assume that* $(y, g) \in \mathbf{S}_{\mathrm{X}} \times \mathbf{S}_{\mathrm{X}^*}$ *satisfies* $g(y) = 1$ *and* y *is a smooth point. Let* $\{(c_n^{(k)})_n | k \in \mathbb{N}\} \subset \mathbf{S}_{\ell^1_+} \cap c_{00}$ *be convex combinations such that*

(i) $\sum_n c_n^{(k)} x_n \xrightarrow{\omega} y$ *as* $k \to \infty$,
(ii) $\sum_n c_n^{(k)} f_n(y) \to 1$ *as* $k \to \infty$.

Then
$$P_k \doteq \sum_n c_n^{(k)} f_n \otimes x_n \xrightarrow{\mathrm{WOT}} g \otimes y \text{ as } k \to \infty,$$
and there are convex combinations $\{(d_n^{(l)})_n | l \in \mathbb{N}\} \subset \mathbf{S}_{\ell^1_+} \cap c_{00}$ *such that*
$$S_l \doteq \sum_n d_n^{(l)} f_n \otimes x_n \xrightarrow{\mathrm{SOT}} g \otimes y \text{ as } l \to \infty.$$

This auxiliary result is applied together with the following elementary, but crucial, observation:
Given $n \in \mathbb{N}$, if $a_1, \ldots, a_n \in (0, \infty)$ are such that $\sum_i a_i = 1$ and $P_1, \ldots, P_n \colon \mathrm{X} \to \mathrm{X}$ are linear operators, then
$$\left|\left|\mathrm{I} - \sum_i a_i P_i\right|\right| \leq \sum_i a_i \left|\left|\mathrm{I} - P_i\right|\right|,$$
where one applies the partition of unity $\mathrm{I} = \sum_i a_i \mathrm{I}$.

Next, we will discuss another topic appearing in [B]. In order to make paper [B] more easily readable, let us recall some concepts related to partial order in Banach spaces. Recall that a *Banach lattice* is a partially ordered Banach space $(\mathrm{X}, \leq)$ such that the following conditions (L1)-(L4) hold:

(L1) $x \leq y$ implies $x + z \leq y + z$ for $x, y, z \in \mathrm{X}$,
(L2) $\alpha x \geq 0$ for every $x \geq 0$ and non negative real α,
(L3) For all $x, y \in \mathrm{X}$ there exist a least upper bound $x \vee y$ and a greatest lower bound $x \wedge y$,
(L4) If $|x| \leq |y|$, then $||x|| \leq ||y||$. Here $|x| = x \vee -x$ for $x \in \mathrm{X}$.

Recall that for a given Banach lattice $(\mathrm{X}, \leq)$ a point $a \in \mathrm{X} \setminus \{0\}$ is an *atom* if $\{v \in \mathrm{X} :\ 0 \leq v \leq |a|\} = \{\lambda|a| :\ 0 \leq \lambda \leq 1\}$. Note that in this event there are *no* disjoint elements $x, y \in \{v \in \mathrm{X} :\ 0 < v \leq |a|\}$. Here the points $x, y \in \mathrm{X}$ are disjoint, $x \perp y$ for short, if $|x| \wedge |y| = 0$.

Consider Banach lattices X and Y. Recall that an operator $T\colon \mathrm{X} \to \mathrm{Y}$ is said to be *disjointness-preserving* if $T(x)$ and $T(y)$ are disjoint whenever $x, y \in \mathrm{X}$ are disjoint. Suppose that a given positive measure μ has an atom and the space $L^p(\mu)$ is convex-transitive for some $1 \leq p \leq \infty,\ p \neq 2$. Then $L^p(\mu)$ is actually 1-dimensional, see [5, Cor.3.5]. We generalize this fact in the following result.

Theorem 9 ([B])**.** *Let* X *be a convex-transitive Banach lattice such that each rotation* $T \in \mathcal{G}_{\mathrm{X}}$ *is disjointness-preserving. If there exists an atom* $x \in \mathbf{S}_{\mathrm{X}}$*, then* X *is in fact* 1*-dimensional.*

In [B] we also study the question whether spaces of the type $\ell^1(\Gamma)$ are somehow typical among the spaces, which admit a big point but are not isometric to a Hilbert space. For this purpose we introduce a concept that to some extent generalizes the notion of *atoms* in Banach lattices to the *general* Banach space setting.

Recall that a continuous linear surjective projection $P\colon \mathrm{X} \to \mathrm{Y}$ is an *isometric reflection projection*, and Y an *isometric reflection subspace* if $\mathrm{I} - 2P \in \mathcal{G}_{\mathrm{X}}$. We say that $u \in \mathbf{S}_{\mathrm{X}}$ is a *strong atom* if the following conditions hold:

(1) $[u]$ is an isometric reflection subspace.
(2) For all closed subspaces $\mathrm{Y} \subset \mathrm{X}$ and all isometric reflection projections $P\colon \mathrm{X} \to \mathrm{Y}$ either $P(u) = u$ or $P(u) = 0$.

Let us consider some basic examples of strong atoms. Suppose that X is a σ-complete Banach lattice and $u \in \mathbf{S}_{\mathrm{X}}$ is a strong atom. Then u is an atom also in the classical sense. This can be seen by using the existence of suitable band projections provided by the σ-completeness, see [25, p.8]. The canonical unit basis vectors e_n of ℓ^p are strong atoms if $1 \leq p \leq \infty$ and $p \neq 2$. This is due to the characterization (1) of rotations of ℓ^p. Moreover, in a Hilbert space H, $\dim(\mathrm{H}) \geq 2$, there are no strong atoms $u \in \mathbf{S}_{\mathrm{H}}$.

Theorem 10 ([B])**.** *Let* X *be a Banach space and* $u \in \mathbf{S}_{\mathrm{X}}$ *a strong atom, which is also a big point. Then* $\mathrm{X} = \ell^1(\Gamma)$ *isometrically for some set* Γ.

Let us recall some concepts that we used above. A subset $A \subset \mathrm{X}$ is called *order bounded* (from above) if there exists $z \in \mathrm{X}$ such that $z \geq x$ for $x \in A$. The Banach lattice X is said to be *σ-complete* if there exists a least upper bound $\bigvee A$ for each countable order bounded subset $A \subset \mathrm{X}$. Here $\bigvee A \in \mathrm{X}$ is defined as the element, which satisfies the following conditions: (1) $\bigvee A \geq x$ for all $x \in A$, (2) if $z \geq x$ holds for some $z \in \mathrm{X}$ and all $x \in A$, then $z \geq \bigvee A$.

Recall also that a subspace $\mathrm{Y} \subset \mathrm{X}$ is an *ideal* if $|y| \in \mathrm{Y}$ whenever $|y| \leq |x|$ and $|x| \in \mathrm{Y}$. An ideal Y is a *band*, if for every subset $A \subset \mathrm{Y}$ such that $\bigvee A \in \mathrm{X}$ exists, it follows that $\bigvee A \in \mathrm{Y}$. A continuous linear projection $P\colon \mathrm{X} \to \mathrm{Y}$ onto a band Y is called a *band projection* if $\mathrm{Ker}(P) = \{x \in \mathrm{X} : |x| \wedge |y| = 0 \text{ for all } y \in \mathrm{Y}\}$.

Summary of Paper C

In [19] the almost transitivity of the Bochner spaces $L^p(\mu, \mathrm{X})$, where $p < \infty$, is studied. If $L^p(\mu)$ and X are almost transitive, then it follows that $L^p(\mu, \mathrm{X})$ is almost transitive. In [C] we prove an analogous result in the convex-transitive setting. Write $L^p(\mathrm{X}) = L^p([0,1], \mathrm{X})$. Denote by $L^\infty_s(\mathrm{X})$ the closed linear span of the simple functions in $L^\infty(\mathrm{X})$. One may show that this subspace of $L^\infty(\mathrm{X})$ consists of essentially compactly valued functions, it is non-separable if $\dim(\mathrm{X}) > 0$, and $L^\infty_s(\mathrm{X}) \subsetneq L^\infty(\mathrm{X})$ if $\dim(\mathrm{X}) = \infty$.

Theorem 11 ([C]). *If* X *is a convex-transitive Banach space, then* $L^\infty_s(\mathrm{X})$ *and* $L^p(\mathrm{X})$, *for* $1 \leq p < \infty$, *are convex-transitive.*

As background information we mention that the rotations of $L^\infty(\mathrm{X})$ can be described as follows [17]: If $\mathrm{X} \neq \mathrm{Y} \oplus_\infty \mathrm{Z}$ for any non-trivial Banach spaces Y and Z, then $T \in \mathcal{G}_{L^\infty(\mathrm{X})}$ if and only if

$$T(f)(s) = \sigma(s)\Phi(f)(s), \quad \text{for } f \in L^\infty(\mathrm{X})$$

holds for a.e. $s \in [0,1]$. Here $\sigma\colon [0,1] \to \mathcal{G}_\mathrm{X}$ is strongly measurable. The rotation $\Phi \in \mathcal{G}_{L^\infty(\mathrm{X})}$ above is obtained from a Boolean isomorphism $\psi\colon \Sigma\backslash_m \to \Sigma\backslash_m$ by extending the transformation $g \mapsto g \circ \psi$ defined for $g = \sum_{k=1}^\infty \chi_{A_k} x_k$, where (A_n) is a countable measurable partition of $[0,1]$ and $(x_k) \subset \mathrm{X}$ is a bounded sequence.

Taking another direction, we prove in [C] the following result, which also is related to vector-valued versions of abstract M-spaces (see [25, p.14] for the definition).

Theorem 12 ([C]). *If* H *is an infinite-dimensional Hilbert space and* L *is a locally compact Hausdorff space such that* $C_0(L)$ *is convex-transitive, then* $C_0(L, \mathrm{H})$ *is convex-transitive.*

This result can be viewed as a convex-transitive analogue of the following one due to Aizpuru and Rambla [1]:

Theorem 13. *Let* L *be a locally compact* σ*-compact Hausdorff space such that* $C_0(L, \mathbb{C})$ *is almost transitive. If* H *is a Hilbert space with* $\dim(\mathrm{H}) \geq 2$, *then* $C_0(L, \mathrm{H})$ *is almost transitive.*

Moreover, we provide some concrete examples of convex-transitive spaces in [C] by suitably modifying spaces of the type $\ell^\infty(\Gamma)$. Recall that ℓ^∞ itself is not convex-transitive.

We also provide the following concrete example of an almost transitive space.

Theorem 14 ([C]). *The hyperplane* $\{x \in L^1 |\ \int_0^1 x(t)\ \mathrm{d}t = 0\} \subset L^1$ *is almost transitive.*

Finally we discuss in [C] the heredity of transitivity in non-separable spaces to subspaces of smaller density. Cabello [9, Cor.1.3] showed that each transitive space contains a separable almost transitive subspace (see also [7, Thm. 2.24]). In [C] we prove, for example, that one can always pass to a suitable transitive subspace of density character at most $2^{\aleph_0}$.

Theorem 15 ([C]). *Let* X *be a transitive space and let* $\mathrm{Y} \subset \mathrm{X}$ *be a subspace with* $\mathrm{dens}(\mathrm{Y}) \leq 2^{\aleph_0}$. *Then there exists a closed subspace* $\mathrm{Z} \subset \mathrm{X}$ *such that*

(1) $\mathrm{Y} \subset \mathrm{Z}$,
(2) $\mathrm{dens}(\mathrm{Z}) \leq 2^{\aleph_0}$,
(3) Z *is transitive with respect to the subgroup* $\mathcal{T} = \{T \in \mathcal{G}_{\mathrm{X}} |\ T(\mathrm{Z}) = \mathrm{Z}\}$ *of* $\mathcal{G}_{\mathrm{X}}$.

Let us recall some general concepts applied in [C] for the sake of completeness of this thesis. Let Γ be a set and $\mathcal{P}(\Gamma)$ its power set. Then $\mathcal{F} \subset \mathcal{P}(\Gamma)$ is a *filter* (on Γ) if it satisfies the following conditions:

(F1) $\emptyset \notin \mathcal{F},\ \Gamma \in \mathcal{F}$
(F2) If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.
(F3) If $A \in \mathcal{F}$ and $A \subset B \subset \Gamma$, then $B \in \mathcal{F}$.

If a filter $\mathcal{F}$ is maximal, that is, for each $A \subset \Gamma$ either $A \in \mathcal{F}$ or $\Gamma \setminus A \in \mathcal{F}$, then $\mathcal{F}$ is called an *ultrafilter*. Ultrafilters are usually denoted by $\mathcal{U}$ in the literature on Banach space.

Let (T, τ) be a topological space, $(x_n) \subset T$ a sequence, $x \in T$ and $\mathcal{F}$ a filter on $\mathbb{N}$. If $\{n \in \mathbb{N} |\ x_n \in U\} \in \mathcal{F}$ for each neighbourhood $U \in \tau$ of x, then x_n converges to x following $\mathcal{F}$ and we denote this by

$$x = \lim_{\mathcal{F}} x_n.$$

If T is a non-empty compact Hausdorff space, $\mathcal{U}$ is an ultrafilter on $\mathbb{N}$ and $(y_n) \subset T$ is any sequence, then it is well-known that there is a unique point $y \in T$ such that $\lim_{\mathcal{U}} y_n = y$.

Let (X_n) be a sequence of Banach spaces and consider the direct sum $\bigoplus_{n\in\mathbb{N}}^{\infty} \mathrm{X}_n$ taken in the ℓ^∞-sense. Consider the natural closed subspace

$$N_{\mathcal{U}} = \{(x_n) \in \bigoplus_{n\in\mathbb{N}}^{\infty} \mathrm{X}_n |\ \lim_{\mathcal{U}} x_n = 0\} \subset \bigoplus_{n\in\mathbb{N}}^{\infty} \mathrm{X}_n.$$

The space

$$\prod_{\mathcal{U}} \mathrm{X}_n \doteq \bigoplus_{n\in\mathbb{N}}^{\infty} \mathrm{X}_n \Big/ N_{\mathcal{U}}$$

is called the *ultraproduct* of (X_n). If $\mathrm{X}_n = \mathrm{X}$ for $n \in \mathbb{N}$ above, then the ultraproduct $\prod_{\mathcal{U}} \mathrm{X}_n$ is called an *ultrapower* of X and it is denoted by $\mathrm{X}_{\mathcal{U}}$.

We refer to [12] for background on ordinals, which are applied in paper [C]. We will not construct the ordinals here but let us recall some of their essential properties (also available in [12]). The class of ordinals is well-ordered. This means that the ordinals are linearly ordered and each non-empty set of ordinals has a (unique) least element. Each ordinal α is a set, and if β is an ordinal, then $\alpha \subset \beta$ if and only if $\alpha \leq \beta$.

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